

Asymptotic analysis of the Skyrmed monopole

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Abstract

We consider a variant of the Georgi-Glashow model in the BPS limit, augmented by a higher derivative Skyrme-like term, which is the simplest YMH model that can support monopole bound states. The spherically symmetric solutions are studied with a combination of analytic and numerical techniques, which strongly suggest that the solutions converge to a finite energy configuration in the limit of infinite coupling of the Skyrme-like term.

1 Introduction

The asymptotic analysis for the *unit* charge 't Hooft-Polyakov monopole [1, 2] was carried out long ago by Kirkman and Zachos [3] and by Gardner [4], and, has recently been elaborated by Forgács *et. al.* [5] by providing a high precision numerical analysis of the problem. In contrast to the Prasad-Sommerfield monopole [6] which is evaluated in closed form, the 't Hooft-Polyakov monopole can be evaluated only numerically since it is a solution to the Georgi-Glashow model that exhibits a symmetry breaking Higgs self interaction potential. The existence of this numerically evaluated solution is underpinned by the purely analytic proof of existence given by Tyupkin, Fate'ev and Schwarz [7], but this does not shed any light on the behaviour of the solution as a function of the strength λ of the Higgs potential term. Numerical studies of this λ dependence, in [8] for the spherically symmetric case and in [9] for the axially symmetric case, reveal in particular that as $\lambda \rightarrow \infty$ the energy of the monopole asymptotes to a finite value. It is this behaviour on λ found numerically that is underpinned by the analytic analysis of [3].

It is our intention in this short note to supply an asymptotic analysis similar to that of [3], for the *unit* charge monopole of a variant of the Georgi-Glashow model characterised by the addition of a Skyrme like term in terms of the covariant derivatives of the Higgs field. The model is the $SO(3)$ Higgs model

$$\mathcal{H} = \frac{1}{8} \left(\frac{1}{4} |F_{ij}^{ab}|^2 + |D_i \phi^a|^2 + \frac{1}{4} \kappa |D_{[i} \phi^a D_{j]} \phi^b|^2 \right) \geq 4\pi \varrho_1, \quad (1)$$

κ giving the strength of the coupling of the Skyrme like term, and κ having the dimension of L^4 . The lower bound on the right hand side is the usual monopole charge density

$$\varrho_1 = \frac{1}{16} \varepsilon_{aa'b} \varepsilon^{ijk} F_{ij}^{aa'} D_k \phi^b \quad (2)$$

featuring a Skyrme like term *in lieu* of the Higgs symmetry breaking potential in the Georgi-Glashow model. This monopole, which was described in [10] as a Skyrmed monopole, is a solution to a model that was distilled from a rather more involved set of models [11, 12], the latter being designed to support monopoles

with both mutually attracting and repelling phases. (The Skyrmed monopole of [10] supports bound states which are axially symmetric for charges higher than 2, in contrast to Skyrmions of charges 3 and higher which instead exhibit Platonic symmetries [13].)

In contrast to the 't Hooft-Polyakov monopole for which there is an analytic proof of existence [7], there are no such proofs for the monopoles of the various generalised models [11, 12, 10]. This is because of the presence of higher order Skyrme like terms, and the only known solutions are those constructed numerically. Here we will supply an asymptotic analysis analogous to that of [3], for the *unit* charged Skyrmed monopole [10] which is the simplest such example available. Due to the considerably more complex structure of the equations here, it is very difficult to perform the promised asymptotic analysis using purely analytic method, and instead we present a combination of both analytic and numerical analysis. (Similar techniques were used in [14], in the context of the $SO(3)$ gauged Skyrmion [15, 16].)

In the next two sections we present the asymptotic and the numerical analyses, respectively, followed by a brief summary of our result.

2 Asymptotic analysis

Since we are restricting to the charge-1 monopole of (1), we impose the usual spherically symmetric Ansatz

$$A_i^{[aa']} = \frac{1 - w(r)}{r} \hat{x}^a \delta^{a'}, \quad \phi^a = \eta h(r) \hat{x}^a \quad (3)$$

where the brackets $[ab]$ imply antisymmetrisation, \hat{x}^a is the unit position vector and η is the dimensionful VEV of the Higgs field. Imposing (3), the residual one dimensional static Hamiltonian is

$$H = w'^2 + \frac{(w^2 - 1)^2}{2r^2} + \frac{1}{2} r^2 h'^2 + w^2 h^2 + \kappa w^2 h^2 \left(2h'^2 + \frac{w^2 h^2}{r^2} \right), \quad (4)$$

having rescaled $r \rightarrow \eta r$ and $\kappa \rightarrow \eta \kappa$ so that both the rescaled radial variable r and the rescaled Skyrme coupling κ in (4) are dimensionless. The corresponding equations for w and h are

$$\begin{aligned} w'' + \frac{w(1 - w^2)}{r^2} - wh^2 &= 2\kappa wh^2 \left((h')^2 + \frac{w^2 h^2}{r^2} \right), \\ r^2 h'' + 2rh' - 2w^2 h &= 4\kappa w^2 h \left((h')^2 + \frac{w^2 h^2}{r^2} \right) - 4\kappa (w^2 h^2 h'' + 2w^2 h(h')^2 + 2ww'h^2 h'), \end{aligned} \quad (5)$$

Due to their complex structure we are only able to extract from the equations some asymptotic information near the origin, at infinity, and for $\kappa \rightarrow \infty$. The asymptotic analysis will then be complemented by the numerical results. For $r \rightarrow 0$, we have $w \rightarrow 1$ and $h \rightarrow 0$. A dominant balance analysis gives

$$w = 1 + w_2 r^2 + w_4 r^4 + O(r^6), \quad h = h_1 r + h_3 r^3 + O(r^5)$$

By induction we see that the asymptotic expansions contain only even or odd powers of r for w and h , respectively. For $\kappa = 0$, the Prasad-Sommerfield solution yields $w_2 = -\frac{1}{6}$ and $h_1 = \frac{1}{3}$. For nonzero κ , w_2 and h_1 have to be determined numerically. The coefficients for the next highest order are

$$w_4 = \frac{3}{10} w_2^2 + \frac{1}{10} (1 + 4\kappa h_1^2) h_1^4, \quad h_3 = \frac{2 - 4\kappa h_1^2}{5 + 20\kappa h_1^2} w_2 h_1,$$

and all other coefficients can be calculated recursively.

For $r \rightarrow \infty$, we have $w \rightarrow 0$ and $h \rightarrow 1$. Here the dominant balance analysis leads to an exponential fall-off for w and to

$$r^2 h'' + 2rh' = 0$$

for the leading term in h . We therefore have

$$h = 1 - \frac{q}{r} + O(\alpha(r)e^{-2r}), \quad w = \beta(r)e^{-r} + O(\gamma(r)e^{-2r}) \quad (6)$$

Using again induction, we see that the asymptotic expansion at infinity for h contains only even powers of e^{-r} , whereas the asymptotic expansion for w contains the odd powers of e^{-r} . The coefficient functions in front of the exponential functions, starting with $\alpha(r)$ and $\beta(r)$, are polynomially bounded. For $\kappa = 0$, we have $q = 1$ and $\beta = 2r$. For nonzero κ , $\beta(r)$ satisfies the equation

$$\beta''(r) - 2\beta'(r) + \frac{1}{r^2}\beta(r) + \left(\frac{2q}{r} - \frac{q^2}{r^2}\right)\beta(r) = 2\kappa \left(\frac{q^2}{r^4} - \frac{2q^3}{r^5} + \frac{q^4}{r^6}\right)\beta(r)$$

and therefore

$$\beta(r) = p r^q \left(1 + \frac{q-1}{2r} + O\left(\frac{1}{r^2}\right)\right)$$

3 Dependance on κ

To study the dependence of the energy

$$E = \int_0^\infty H dr$$

on κ , we calculate

$$\begin{aligned} \frac{dE}{d\kappa} &= \int_0^\infty \left(\frac{\partial H}{\partial \kappa} + \frac{\partial w}{\partial \kappa^2} \frac{\partial H}{\partial w} + \frac{\partial w'}{\partial \kappa} \frac{\partial H}{\partial w'} + \frac{\partial h}{\partial \kappa} \frac{\partial H}{\partial h} + \frac{\partial h'}{\partial \kappa^2} \frac{\partial H}{\partial h'} \right) dr \\ &= \int_0^\infty w^2 h^2 \left(2(h')^2 + \frac{w^2 h^2}{r^2} \right) dr \stackrel{\text{def.}}{=} E_{sk} > 0 \end{aligned} \quad (7)$$

Here we have used integration by parts, the equations for w and h , and the boundary conditions. We see that the energy increases with κ . We also see that, if the energy is bounded as $\kappa \rightarrow \infty$ (as suggested by the numerical results reported in the next section), E_{sk} and therefore the product wh must vanish in this limit. Because of the boundary conditions, w cannot be zero for small r , whereas h cannot vanish for large r . This strongly suggests that h is zero in some interval $(0, r_m)$ and that w is zero in the interval (r_m, ∞) .

In the limit $\kappa \rightarrow \infty$ we therefore expect the following equations to hold,

$$w''(r) + \frac{w(r)(1-w^2(r))}{r^2} = 0, \quad h(r) = 0 \quad (0 < r < r_m), \quad (8)$$

$$w(r) = 0, \quad r^2 h''(r) + 2rh'(r) = 0 \quad (r_m < r < \infty) \quad (9)$$

with boundary conditions

$$w(0) = 1, \quad w(r_m) = 0, \quad h(r_m) = 0 \quad \text{and} \quad h \rightarrow 1 \quad \text{as} \quad r \rightarrow \infty$$

The solution of equation (9) is

$$h(r) = 0 \quad (0 < r < r_m), \quad h(r) = 1 - \frac{r_m}{r} \quad (r_m < r < \infty)$$

The solutions to equation (8) have been studied a long time ago [17], but not to the same extent as other special functions. One property we can deduce immediately from equation (8) is that, because w'' is negative for $r < r_m$, $w'(r_m)$ cannot be zero.

Denoting the solution of Eqs. (8),(9) with the appropriate boundary condition at $r = r_m$ by w_∞, h_∞ , the corresponding energy can be obtained easily :

$$E_\infty(r_m) = \int_0^{r_m} \left(w_\infty'^2 + \frac{(w_\infty^2 - 1)^2}{2r^2} \right) dr + \int_{r_m}^\infty \left(\frac{1}{2r^2} + \frac{1}{2} r^2 h_\infty'^2 \right) dr = E_w(r_m) + \frac{1+r_m^2}{2r_m}. \quad (10)$$

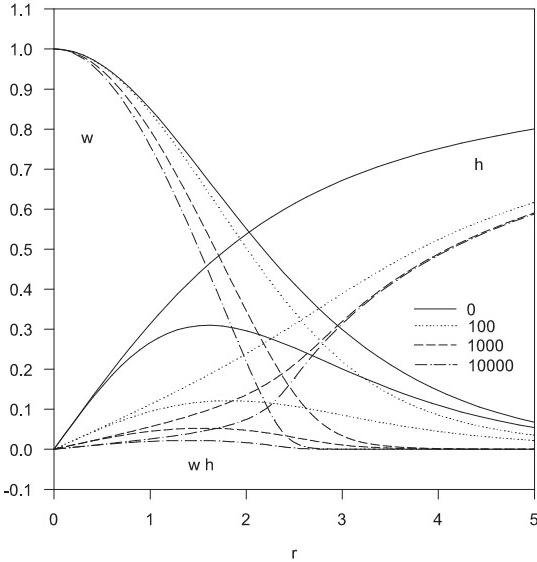


Figure 1: The profile for the solution for several values of κ .

From the second term we see that $E_\infty \rightarrow \infty$ for $r_m \rightarrow 0$ and for $r_m \rightarrow \infty$; so $E_\infty(r_m)$ has a minimum. Solving Eq. (8) numerically for several values of r_m and computing the value $E_\infty(r_m)$, we have determined the local minimum of E_∞ which we find to occur for $r_{m,c} \approx 2.0623$. Furthermore, our result strongly suggests the relation $E_\infty(r_{m,c}) = r_{m,c}$; we have no analytic proof for this to be an identity but it holds within our numerical accuracy i.e. 10^{-4} . The configuration minimizing E_∞ has $w''_\infty(0) \approx -0.656$.

The results of the numerical analysis reported in the next section will strongly confirm that

$$\lim_{\kappa \rightarrow \infty} (w(r), h(r)) = (w_\infty, h_\infty) \quad \text{with} \quad r_m = r_{m,c}. \quad (11)$$

for the solution w, h of Eqs. (5).

4 Numerical analysis

We now discuss the numerical solutions of Eq. (5) for finite κ . In the limit $\kappa = 0$, the classical equations coincide with the equations of the BPS monopole which is known explicitly. To the best of our knowledge, no explicit solution exist for $\kappa > 0$. We have solved equations (5) completed with the boundary conditions,

$$w(0) = 1, \quad h(0) = 0, \quad w(\infty) = 0, \quad h(\infty) = 1 \quad (12)$$

by using a numerical solver [18]. The PBS monopole gets smoothly deformed for $\kappa > 0$. This is illustrated in Fig. 1, where the profiles w, h of the BPS monopole solution are superposed with solutions corresponding to several positive values of κ . For reasons explained in the previous section, we supplemented this figure with the profiles of the product wh .

Several parameters characterizing the solutions, namely the classical energy E , the energy of the Skyrme term E_{sk} (see (7)), the value of q (see (6)), $h_1 \equiv h'(0)$ and $w_2 \equiv w''(0)/2$ are plotted as functions of κ in Fig. 2. The characteristics of the PBS monopole are recovered in the limit $\kappa = 0$, for instance $E = 1$, $q = 1$, $h'(0) = 1/3$, $w''(0) = -1/6$. The natural challenge is to construct numerically the solutions for large values of κ and to confirm that they evolve according to the pattern discussed in the previous section. The quantities w_2, h_1, q, E extracted from our numerical solutions are reported on Fig. 2. The figure shows that they stay finite for $\kappa \gg 1$. In particular for $\kappa \geq 10^6$ we find $q \approx 2.0623$, $w''(0) \approx -0.656$, $h(0) \approx 0$ suggesting

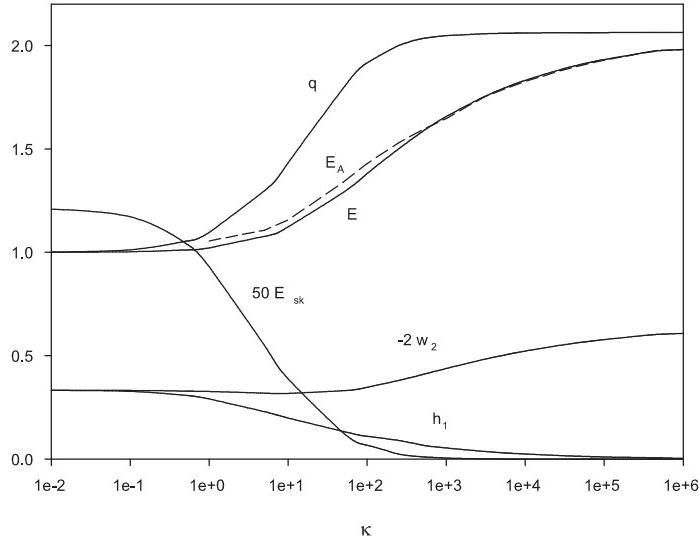


Figure 2: The values of $E, E_{sk}, q, h_1 \equiv h'(0), w_2 \equiv w''(0)/2$ are plotted as functions of κ (solid lines) the values of E_A are represented with the dashed line.

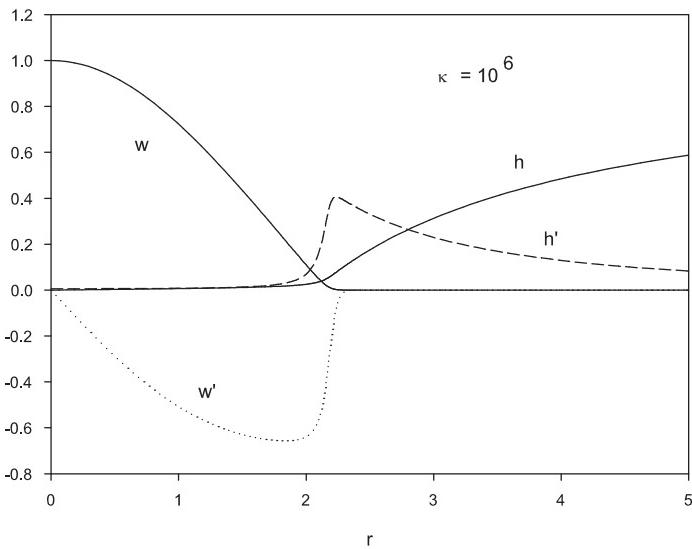


Figure 3: The profile for the solution corresponding to $\kappa = 10^6$

that, in the limit $\kappa \rightarrow \infty$, the solution approaches the configuration (8),(9) which minimizes the energy E_∞ . Our numerical results further indicate that the product wh tends uniformly to the null function in the limit $\kappa \rightarrow \infty$. As anticipated in the previous section, we observe from our numerical solutions that, in the interior region, i.e. for $r \in [0, r_m]$ we have $h(r) \sim 0$, while in the exterior region we have rather $w(r) \sim 0$. Of course the value of r_m cannot be precisely determined for $\kappa < \infty$ but the numerical results obtained for large κ are quite compatible with $r_m = r_{m,c} \approx 2.0623$ and with $r_m = q$. This is illustrated in Fig. 3 for $\kappa = 10^6$ where the functions w, h and the derivatives w', h' are plotted as functions of r . The limit of the solutions of the equations (5) is therefore not differentiable at $r = r_m$, i.e., the equations (5) are singularly perturbed about $1/\kappa = 0$. As a consequence of the absence of differentiability of the limiting solution at an intermediate point of the domain of integration, the numerical analysis becomes involved for increasing κ .

Coming finally to the energy of the solution, the numerical evaluation of E is fully compatible with the fact that the energy stays finite and that $E_{\kappa \rightarrow \infty} = 2.0623$. The convergence of E for $\kappa \rightarrow \infty$ is, however, much slower than the convergence of the parameter q . To argue that this statement is correct, we further evaluate the quantity

$$E_A(\kappa) = \int_0^q \left(w'^2 + \frac{(w^2 - 1)^2}{2r^2} \right) dr + \frac{1+q^2}{2q} . \quad (13)$$

with the numerical profile of $w(r)$. In fact, E_A is an estimation of the energy obtained by assuming $h = 1 - q/r$, $w = 0$ for $r \in [q, \infty]$ and $h = 0$ and the numerical value of $w(r)$ for $r \in [0, q]$. One should expect $E_A(\kappa \rightarrow \infty) = E_\infty(r_{m,c})$ since $q \sim r_{m,c}$ in this limit.

The value E_A is reported on Fig. 2 (see the dashed line); confirming our expectation, we find that the difference $E - E_A$ tends quickly to zero (in fact exponentially fast) for $\kappa \rightarrow \infty$. Interestingly, E_A provides a reasonably good approximation to the energy E even for small values of κ ; for instance we find $E_A/E \sim 1.035$ for $\kappa = 1$.

5 Summary

We have carried out a combination of analytic and numerical analysis for what we refer to as the Skyrmed monopole, which is a finite energy solution to the equations of the model (1). The asymptotic analysis involves the study of a one parameter family of monopole solutions, parametrised by the strength of the coupling of the quartic Skyrme like term. For any value of the coupling constant κ we have given asymptotic expansions of the solutions for $r \rightarrow 0$ and $r \rightarrow \infty$. We have concentrated, however, mainly on the behaviour of the solutions for $\kappa \rightarrow \infty$. The numerical analysis shows that the energy is bounded as $\kappa \rightarrow \infty$, a result for which we have no mathematical underpinning. Given that the energy is bounded, we can however deduce analytically some interesting results. We find in particular that the equations (5) are singularly perturbed about $1/\kappa = 0$. This is reminiscent of Burgers' equation with a small coefficient in front of the second order derivative. When this coefficient goes to zero, the solutions of Burgers' equations tend to a weak solution with a shock, i.e., the limiting weak solution is discontinuous. In our case, the derivative of the limiting weak solution is discontinuous. This analytic result, and the results we get for the values of some typical constants are all supported by our numerical analysis.

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